State Estimation of Linear and Nonlinear Dynamic Systems

Part III: Nonlinear Systems: Extended Kalman Filter (EKF) and Unscented Kalman Filter (UKF)

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Outline

Nonlinear Dynamic Systems

- For the nonlinear model with Gaussian noise

$$x(k+1) = F(x,u) + G(x,u)w$$
$$y(k) = h(x) + v$$

$$w \sim N(0,Q) \qquad v \sim N(0,R) \qquad x(0) \sim N(\overline{x}_0, Q_0)$$

- Consider the linearization at every k

$$\overline{A}(k) = \left.\frac{\partial F(x,u)}{\partial x}\right|_{\widehat{x}(k),u(k)} \qquad \overline{C}(k) = \left.\frac{\partial h(x)}{\partial x}\right|_{\widehat{x}(k),u(k)}$$

$$\overline{G}(k) = G(\widehat{x}(k), u(k))$$

Extended Kalman Filter (EKF) Recursion

- Forecast

$$\widehat{x}^-(k+1) = F(\widehat{x}, u)$$
$$P^-(k+1) = \overline{A}(k)P(k)\overline{A}'(k) + \overline{G}(k)Q\overline{G}'(k)$$
$$\widehat{x}^-(0) = \overline{x}_0 \qquad P^-(0) = Q_0$$

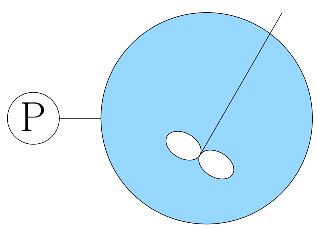
- Correction

$$\widehat{x}(k) = \widehat{x}^-(k) + L(k)(y(k) - h(\widehat{x}^-(k)))$$
$$L(k) = P^-(k)\overline{C}'(k)(\overline{C}(k)P^-(k)\overline{C}'(k) + R)^{-1}$$
$$P(k) = P^-(k) - L(k)\overline{C}(k)P^-(k)$$

Extended Kalman Filter — Remarks

- EKF has a similar recursion in structure to the KF with
  - Mean propagation through the full nonlinear model
  - Covariance propagation through the linearized model
- Resulting error from linearization may cause filter divergence

EKF on Batch Reactor Example



$$\mathrm{A} \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} \mathrm{B} + \mathrm{C}$$

$$2\mathrm{B} \underset{k_{-2}}{\overset{k_2}{\rightleftharpoons}} \mathrm{C}$$

- Estimate the concentrations of A, B, and C
- Model

$$\frac{d}{dt}\begin{bmatrix} c_A \\ c_B \\ c_C \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} k_1 c_A - k_{-1} c_B c_C \\ k_2 c_B^2 - k_{-2} c_C \end{bmatrix}$$

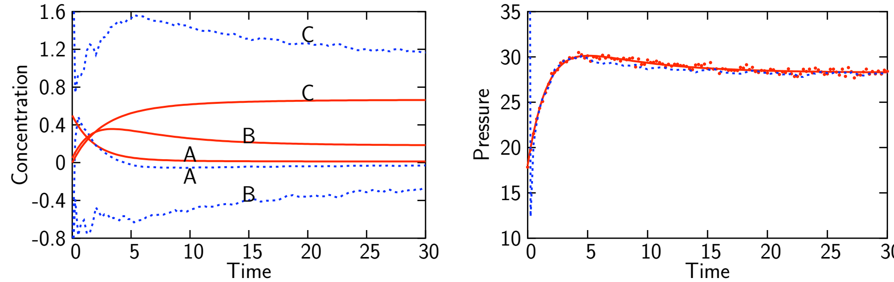
- Measure the total pressure

$$y = RT\left(c_A + c_B + c_C\right)$$

- Poor initial guess

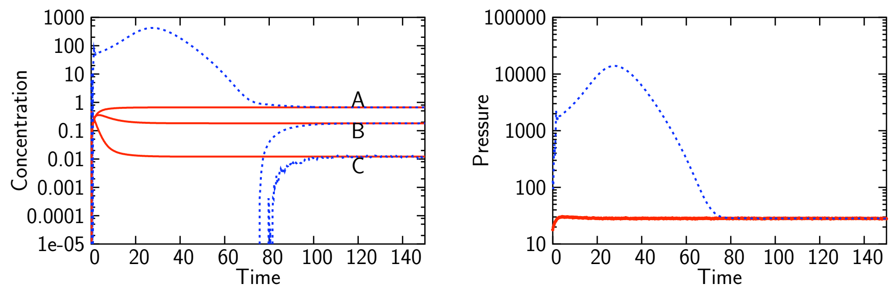
$$x_0 = \begin{bmatrix} 0.5 & 0.05 & 0 \end{bmatrix}^T \text{ vs. } \bar{x}_0 = \begin{bmatrix} 0 & 0 & 4 \end{bmatrix}^T$$

EKF Results



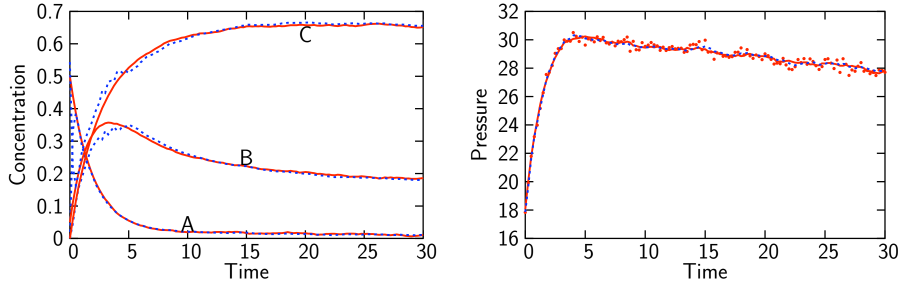
Component	Predicted EKF Steady-State	Actual Steady-State
A	-0.027	0.012
B	-0.238	0.184
C	1.137	0.675

Clipped EKF Results



Clipping of States: $c_j < 0 \rightarrow c_j = 0, \ j = A, B, C$

Constrained MHE Results



State Constraints: $c_j \geq 0$, $j = A, B, C$

Extended Kalman Filter — Assessment

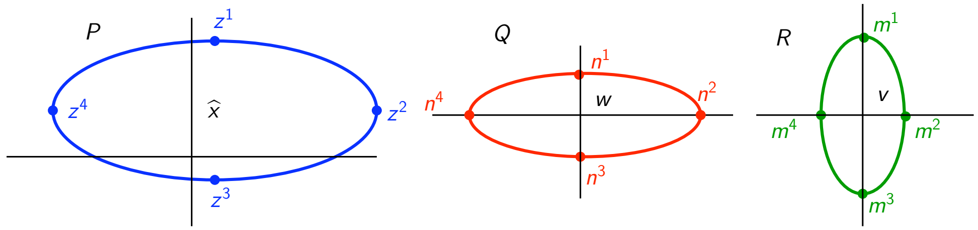
The extended Kalman filter is probably the most widely used estimation algorithm for nonlinear systems.

However, more than 35 years of experience in the estimation community has shown that it is difficult to implement, difficult to tune, and only reliable for systems that are almost linear on the time scale of the updates.

Many of these difficulties arise from its use of linearization.

Julier and Uhlmann (2004).

Unscented Kalman Filter (UKF)



- Given $\hat{x}$ and P, choose sample points, z^i, and weights, w^i, such that

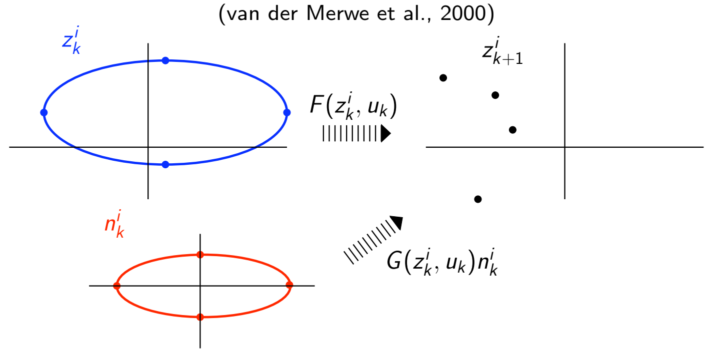
$$\hat{x} = \sum_i w^i z^i \qquad P = \sum_i w^i (z^i - \hat{x})(z^i - \hat{x})'$$

- Similarly, given $w \sim N(0, Q)$ and $v \sim N(0, R)$, choose sample points n^i for w and m^i for v.

UKF Prediction Step

Nonlinear Transformation

(van der Merwe et al., 2000)



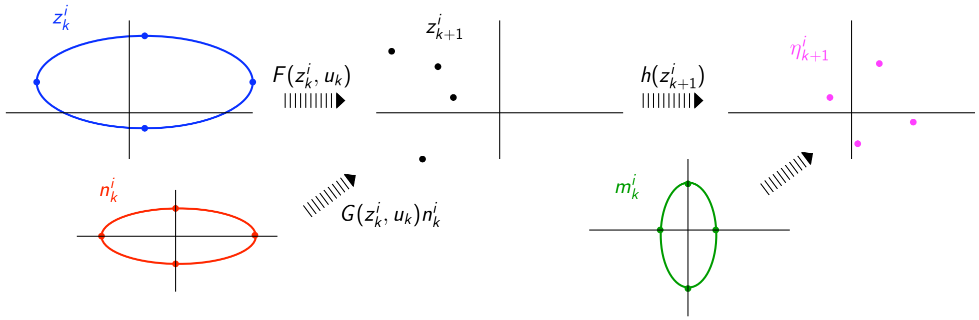
- Propagate sigma points with the nonlinear model

$$z_{k+1}^i = F(z_k^i, u_k) + G(z_k^i, u_k) n_k^i \quad \text{all } i$$

- From these compute the forecast

$$\hat{x}_{k+1}^- = \sum_i w^i z_{k+1}^i \qquad P_{k+1}^- = \sum_i w^i (z_{k+1}^i - \hat{x}_{k+1}^-)(z_{k+1}^i - \hat{x}_{k+1}^-)'$$

UKF Measurement Update



- Measurement forecast:

$$\eta_{k+1}^i = h(z_{k+1}^i) + m_k^i, \qquad \widehat{y}_{k+1}^- = \sum_i w^i \eta_{k+1}^i$$

- Output error: $\mathscr{Y} := y - \widehat{y}^-$

UKF Recursion

- First rewrite the Kalman filter update

$$\begin{aligned}
\widehat{x} &= \widehat{x}^- + L(y - \widehat{y}^-) \\
L &= \underbrace{\mathcal{E}((x - \widehat{x}^-)\mathscr{Y}')}_{P^-C'} \underbrace{\mathcal{E}(\mathscr{Y}\mathscr{Y}')^{-1}}_{(CP^-C'+R)^{-1}} \\
P &= P^- - L \underbrace{\mathcal{E}((x - \widehat{x}^-)\mathscr{Y}')'}_{CP^-}
\end{aligned}$$

- Approximate the two expectations with the sigma point samples

$$\begin{aligned}
\mathcal{E}((x - \widehat{x}^-)\mathscr{Y}') &\approx \sum_i w^i (z^i - \widehat{x}^-)(\eta^i - \widehat{y}^-)' \\
\mathcal{E}(\mathscr{Y}\mathscr{Y}') &\approx \sum_i w^i (\eta^i - \widehat{y}^-)(\eta^i - \widehat{y}^-)'
\end{aligned}$$

UKF Assessment

- Does not linearize at a single point. Samples the nonlinearity at several places ($2n$).
- Computationally efficient.
- Does not require even the Jacobian $\partial F(x, u)/\partial x$ of the model.
- Has been tested on simulation examples, including process control examples (exothermic CSTR, pH). (Romanenko and Castro, 2004; Romanenko et al., 2004).

UKF Assessment (cont'd)

- Attractive alternative if the EKF gives convergence problems or proves difficult to tune.
- Recently published work incorporates constraints in the UKF formulation (Vachhani et al., 2006)
  - Performance not yet compared to other nonlinear and constrained approaches such as
    - Moving Horizon Estimation (optimization based)
    - Particle Filtering (sampling based)

Conclusions

Here we have learned ...

- State estimation approaches for unconstrained nonlinear systems
  - Extended Kalman Filter
    - Example showed EKF divergence
  - Unscented Kalman Filter
    - Attractive alternative if the EKF fails
    - Incorporation of constraints is under development

Further Reading

S. J. Julier and J. K. Uhlmann. Unscented filtering and nonlinear estimation. *Proc. IEEE*, 92(3):401–422, March 2004.

A. Romanenko and J. A. A. M. Castro. The unscented filter as an alternative to the EKF for nonlinear state estimation: a simulation case study. *Comput. Chem. Eng.*, 28 (3):347–355, March 15 2004.

A. Romanenko, L. O. Santos, and P. A. F. N. A. Afonso. Unscented Kalman filtering of a simulated pH system. *Ind. Eng. Chem. Res.*, 43:7531–7538, 2004.

P. Vachhani, S. Narasimhan, and R. Rengaswamy. Robust and reliable estimation via unscented recursive nonlinear dynamic data reconciliation. *J. Proc. Cont.*, 16(10): 1075–1086, December 2006.

R. van der Merwe, A. Doucet, N. de Freitas, and E. Wan. The unscented particle filter. Technical Report CUED/F-INFENG/TR 380, Cambridge University Engineering Department, August 2000.